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Non-linear oscillations of a Hamiltonian system in the case of 3:1 resonance☆

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ABSTRACT

The motion of an autonomous Hamiltonian system with two degrees of freedom near its equilibrium position is considered. It is assumed that, in a certain region of the equilibrium position, the Hamiltonian is an analytic and sign-definite function, while the frequencies of linear oscillations satisfy a 3:1 ratio. A detailed analysis of the truncated system, corresponding to the normalized Hamiltonian is given, in which terms of higher than the fourth order are dropped. It is shown that the truncated system can be integrated in terms of Jacobi elliptic functions, and its solutions describe either periodic motions or motions that are asymptotic to periodic motions, or conventionally periodic motions. It is established, using the KAM-theory methods, that the majority of conventionally periodic motions are also preserved in the complete system. Moreover, in a fairly small neighbourhood of the equilibrium position, the trajectories of the complete system, which are not conventionally periodic, form a set of exponentially small measure. The results of the investigation are used in the problem of the motion of a dynamically symmetrical satellite in the region of its cylindrical precession.

The problem of the behaviour of the trajectories of a dynamical system close to an equilibrium position when there are resonances has been considered from various points of view. In Refs 1-6 the problem of the existence and orbital stability in the linear approximation of the periodic solutions of an autonomous Hamiltonian system with two degrees of freedom originating from an equilibrium position when there are resonances was investigated. A rigorous analysis was carried out of the orbital stability of these periodic solutions in Refs 7-10. The motions of an accelerated system, the Hamiltonian of which is a resonance normal form, containing terms no higher than the fourth power, were considered in Refs 1, 11-13. The phase portraits of the accelerated system were investigated in Refs 14 and 15. A geometrical approach for describing the phase flux of an accelerated system was proposed in Ref. 16. Using methods of local analysis and the KAM-theory, a complete qualitative investigation of the non-linear oscillations of an autonomous Hamiltonian system with two degrees of freedom with resonances of the first, second and third orders was carried out in Refs 8, 9, 17, 18, and also in Ref. 19 for the case of a fourth-order resonance on the assumption that the quadratic part of the Hamiltonian function is sign-variable.

1. Formulation of the problem

Consider the autonomous Hamiltonian system with two degrees of freedom

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2 \tag{1.1}$$

the origin of coordinates $q_j = p_j = 0$ ($j = 1, 2$) of the phase space of which is an equilibrium position. The Hamiltonian $H(q_1, q_2, p_1, p_2)$ is analytic in a certain neighbourhood of the origin of coordinates, while the eigenvalues $\pm i\lambda_j$ ($\lambda_j > 0, j = 1, 2$) of the matrix of the linearized system are pure imaginary. It is assumed that the quadratic part of the Hamiltonian is a sign-definite function, while the frequencies of linear oscillations are in the ratio of 3:1, i.e., there is fourth-order resonance $\lambda_1 = 3\lambda_2$.

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In this case we can introduce canonical variables q_i, p_i so that we can write the Hamiltonian of system (1.1) in the following normal form[14,20]

$$H = \frac{1}{2}\lambda_1(q_1^2+p_1^2)+\frac{1}{2}\lambda_2(q_2^2+p_2^2)+$$
$$+\frac{1}{4}c_{20}(q_1^2+p_1^2)^2+\frac{1}{4}c_{11}(q_1^2+p_1^2)(q_2^2+p_2^2)+\frac{1}{4}c_{02}(q_2^2+p_2^2)^2-$$
$$-\frac{b}{4}[q_2^2(q_1q_2+3p_2p_1)-p_2^2(p_2p_1+3q_1q_2)]+O_6 \tag{1.2}$$

where $c_{20}, c_{11}, c_{02}, b\,(b \geq 0)$ are constant coefficients. We will denote by O_6 the series in powers of the variables q_j, p_j $(j = 1, 2)$ which converges in a certain neighbourhood of the equilibrium position and which begins with terms no less than the sixth power.

We will introduce a new independent variable $\tau = \lambda_2 t$ and make the replacement of canonical variables $q_j, p_j \to x_j, y_j$ using the formulae

$$q_i = \sqrt{\frac{\varepsilon\lambda_2}{b}}x_i, \quad p_i = \sqrt{\frac{\varepsilon\lambda_2}{b}}y_i$$

where ε is a small quantity. Here and henceforth it is assumed that $b \neq 0$.

The Hamiltonian takes the following form in the variables x_j, y_j $(j = 1, 2)$

$$H = \frac{3}{2}(x_1^2+y_1^2)+\frac{1}{2}(x_2^2+y_2^2)+$$
$$+\frac{1}{4}\varepsilon\{a_{20}(x_1^2+y_1^2)^2+a_{11}(x_1^2+y_1^2)(x_2^2+y_2^2)+a_{02}(x_2^2+y_2^2)^2-$$
$$-x_2^2(x_1x_2+3y_2y_1)+y_2^2(y_2y_1+3x_1x_2)\}+O(\varepsilon^2), \quad a_{ij} = \frac{c_{ij}}{b}, \quad i+j = 2 \tag{1.3}$$

With the assumption that the quadratic part of the Hamiltonian is sign-definite, the presence of a resonance in the system cannot lead to instability of the equilibrium position $q_j = p_j = 0$ $(j = 1, 2)$. However, in the case of resonances the structure of the phase space in the neighbourhood of the equilibrium position differs considerably from the general non-resonance case.

Below we investigate the qualitative nature of the behaviour of the system trajectories (1.1) in the neighbourhood of the equilibrium position. We will use a method similar to that employed earlier in Refs 8, 9, 17-19.

2. Investigation of the truncated system

We will consider the motion of the truncated system with Hamiltonian

$$H_* = 3r_1+r_2+\varepsilon\{a_{20}r_1^2+a_{11}r_1r_2+a_{02}r_2^2+r_2^{3/2}\sqrt{r_1}\cos(\varphi_1-3\varphi_2)\} \tag{2.1}$$

which is obtained from the Hamiltonian (1.3) of the complete system, if terms of order higher than ε in the complete system are neglected and the following canonical replacement of variables is made

$$x_i = \sqrt{2r_i}\sin\varphi_i, \quad y_i = \sqrt{2r_i}\cos\varphi_i$$

In addition to the energy integral $H_* = \text{const}$, the truncated system has one other first integral $3r_1 + r_2 = \text{const}$ and can be integrated in quadratures. Below we will show that its general solution can be obtained explicitly in terms of elliptic functions and elliptic integrals.

We will make one more canonical replacement of variables

$$\psi = \varphi_2, \quad \theta = \varphi_1-3\varphi_2, \quad J = 3r_1+r_2, \quad R_2 = r_1 \tag{2.2}$$

Then the Hamiltonian of the truncated system takes the form

$$H_* = J+\varepsilon\{a_{02}J^2+\sigma JR_2+\mu R_2^2+\sqrt{R_2}(J-3R_2)^{3/2}\cos\theta\} \tag{2.3}$$

where $\sigma = a_{11} - 6a_{02}$, $\mu = a_{20} - 3a_{11} + 9a_{02}$.

In the new variables the additional first integral can be written in the form $J = J_0 = \text{const}$. The equations for the variables θ and R_2

$$\frac{d\theta}{d\tau} = \varepsilon\left\{\frac{1}{2}\frac{\sqrt{J_0-3R_2}(J_0-12R_2)\cos\theta}{\sqrt{R_2}}+\sigma J_0+2\mu R_2\right\}$$
$$\frac{dR_2}{d\tau} = \varepsilon\sqrt{R_2}(J_0-3R_2)^{3/2}\sin\theta \tag{2.4}$$

form a closed system of differential equations, in which J_0 plays the role of the parameter that describes the motion of the canonical system with one degree of freedom. The Hamiltonian of this system has the form

$$F = \varepsilon(\sigma J_0R_2+\mu R_2^2+\sqrt{R_2}(J_0-3R_2)^{3/2}\cos\theta) \tag{2.5}$$

and is its first integral $F = \varepsilon h = \text{const}$.

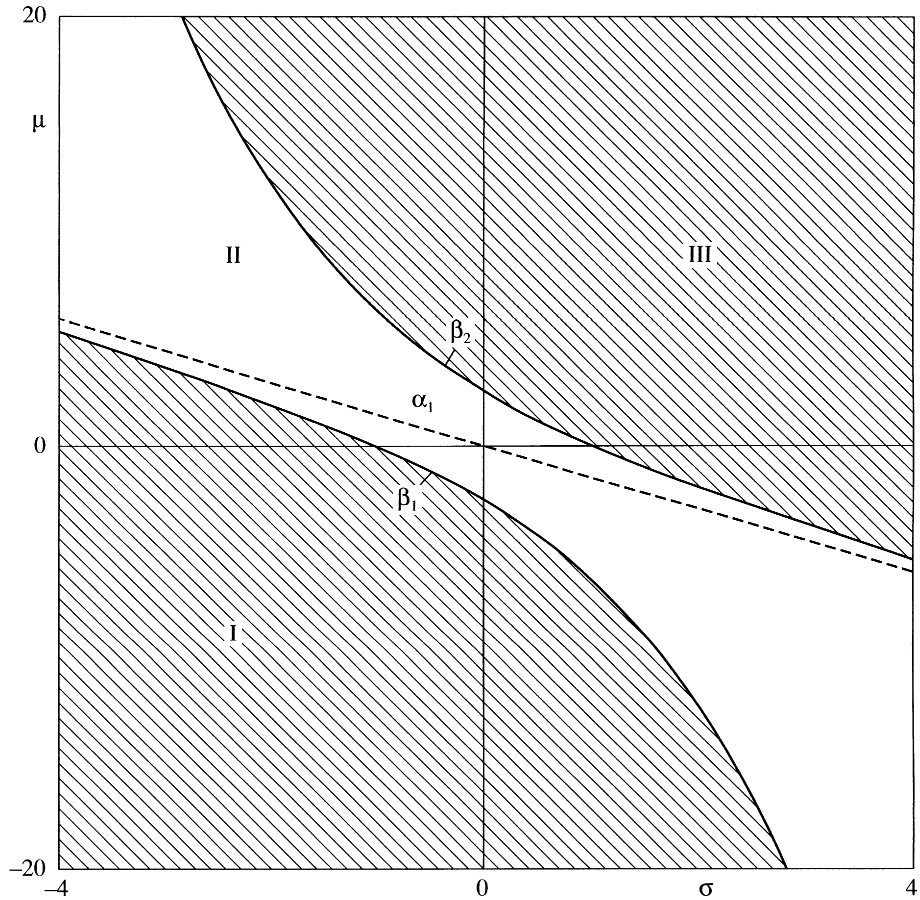

Fig. 1.

System (2.4) has equilibrium positions, defined by the formulae

$$\theta = \frac{\pi}{2}[1 - \mathrm{sign}((12x_* - 1)(\sigma + 2\mu x_*))], \quad R_2 = x_* J_0 \tag{2.6}$$

where x_* is the real root of the equation

$$(432 + 16\mu^2)x^3 + (16\mu\sigma - 216)x^2 + (4\sigma^2 + 27)x - 1 = 0 \tag{2.7}$$

Here, in view of the obvious inequality $J \geq 3R_2$ the root x_* must belong to the interval (0, 1/3].

We will show that the real roots of Eq. (2.7) always lie in this interval. With this aim in view we will write the left-hand side of this equation in the form

$$f(x) = 4x(2x\mu + \sigma)^2 + (3x - 1)(12x - 1)^2 \tag{2.8}$$

It is easy to see that if $x \leq 0$, then for any values of the parameters σ and μ the function $f(x)$ is negative. If $x > 1/3$, then for any σ and μ the function $f(x)$ only takes positive values. Hence, $f(x)$ can only vanish in the interval (0, 1/3]. This indicates that a position of equilibrium of system (2.4) corresponds to each real root of Eq. (2.7).

The results of an analysis of the roots of Eq. (2.7) are shown in the form of a diagram in the σ, μ plane in Fig. 1. In region I, Eq. (2.7) has one real root, which we will henceforth denote by $x^{(1)}$. On the boundary curve β_1, in addition to $x^{(1)}$, Eq. (2.7) also has a multiple real root. This root can be split into real roots $x^{(2)}$ and $x^{(3)}$ on transferring from the boundary curve β_1 to the region II. Hence, in region II Eq. (2.7) has three real roots $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$. The roots $x^{(1)}$ and $x^{(3)}$ merge on the curve of β_2, forming a multiple real root which disappears on transferring into region II. In region III Eq. (2.7) has a unique real root $x^{(2)}$.

The boundary curves β_1 and β_2 are given by the equation

$$48\sigma^2 + 648\sigma^2 + 32\sigma^3\mu + 648\sigma\mu + 108\mu^2 - 729 = 0 \tag{2.9}$$

which is obtained from the condition for multiple roots of Eq. (2.7) to exist. The curves of β_1 and β_2 can also be specified in the following parametric form

$$\sigma = \pm\frac{3(6s - 1)}{\sqrt{16s - 48s^2}}, \quad \mu = \mp\frac{72s^2 - 12s - 1}{2s\sqrt{16s - 48s^2}} \tag{2.10}$$

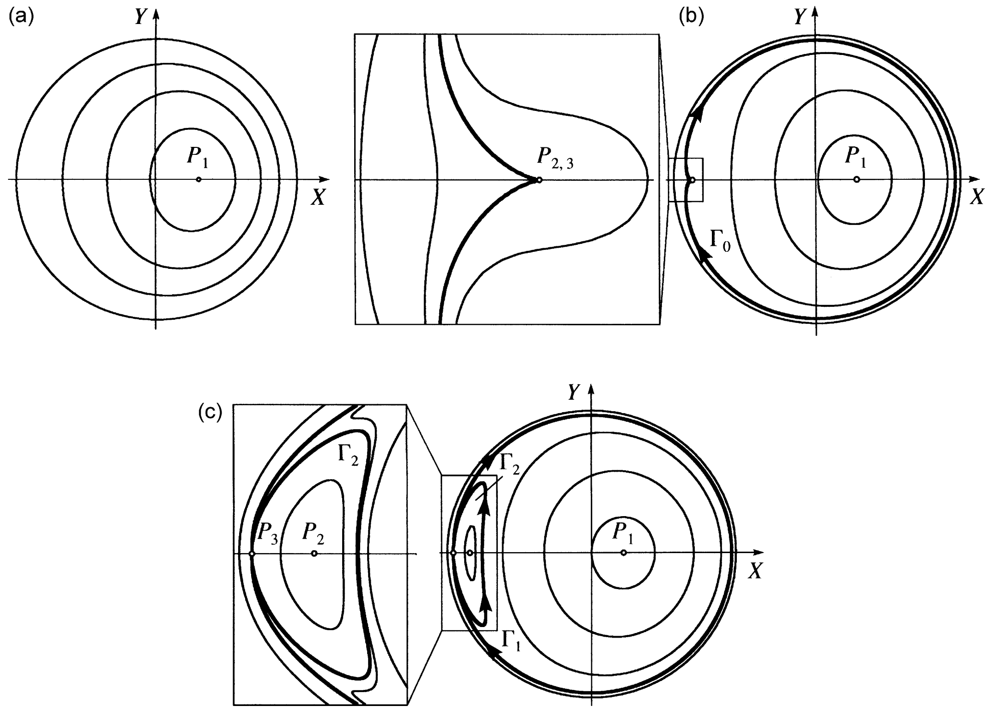

Fig. 2.

where s takes values from the interval (0, 1/3] and is actually a multiple root of Eq. (2.7). The lower sign in formulae (2.10) denotes the curve β_1 and the upper sign denotes the curve β_2. We will have more to say below regarding the straight line α_1, shown in Fig. 1.

The families of long-period motions of the truncated system, the period of which with respect to τ is close to 2π, correspond to equilibrium positions (2.6) of system (2.4). These motions also exist in the complete system with Hamilton (1.3) (see Ref.6).

In addition to the equilibrium positions, system (2.4) has a particular solution of the form

$$R_2 = \frac{J_0}{3}, \quad \theta = \varepsilon\frac{J_0}{3}(3\sigma + 2\mu)\tau + \theta_0 \tag{2.11}$$

where θ_0 is a constant quantity which takes any value from the interval $[0, 2\pi)$. Solution (2.11) corresponds to a family of so-called short-period motions of the truncated system, the period of which is close to $2\pi/3$ with respect τ. The existence of short-period motions in the complete system follows from Lyapunov's theorem on the holomorphic integral.

The structure of the phase space of the truncated system can be analysed using phase portraits, constructed for fixed values of J_0 and which describe the behaviour of the trajectories of reduced system (2.4). Phase portraits, represented in the plane of the variables

$$X = \sqrt{2R_2}\cos\theta, \quad Y = \sqrt{2R_2}\sin\theta$$

are shown in Figs 2 and 3. The points P_1, P_2 and P_3 denote the equilibrium positions of the reduced system (2.4), corresponding to the roots $x^{(1)}, x^{(2)}, x^{(3)}$ of Eq. (2.7) respectively. The region of possible motion is bounded by the phase curve, which is a circle with centre at the origin of coordinates and radius $\sqrt{2J_0/3}$, the motion along which is described by solutions (2.11).

We will give some additional commentaries on Figs. 2 and 3. The phase portrait is shown in Fig. 2a for values of the parameters σ and μ from subregion I (see Fig. 1). For boundary curve β_1 the corresponding phase portrait is shown in Fig. 2b. In Fig. 2c we show the phase portrait for values of the parameters lying in subregion II below the straight line α_1, and for values of the parameters corresponding to this straight line the phase portrait is shown in Fig. 3a. In Fig. 3b we show the phase portrait corresponding to values of the parameters from that part of subregion II which is above the straight line α_1. The phase portrait corresponding to the boundary curve β_2 is shown in Fig. 3c. In Fig. 3d we show the phase portrait for values of the parameters from subregion III.

Changes in the phase portraits that occur when the parameters change are clearly related to bifurcation of the equilibrium positions. There is one stable equilibrium position P_1 (Fig. 2a) for values of the parameters σ and μ from subregion I. When the parameters take values on the boundary curve β_1, one more equilibrium position $P_{2,3}$ appears, which is unstable. On transferring into subregion II branching of $P_{2,3}$ into a stable equilibrium position P_2 and an unstable equilibrium position P_3 occurs. The changes in the phase portraits, corresponding to this bifurcation, are shown in Fig. 2b and c. If the parameters of the reduced system (2.4) are changed so that the point in the σ, μ plane corresponding to them, moving from boundary curve β approach the straight line α_1, the stable equilibrium position P_2 in the phase plane (Fig. 2c) of system (2.4) is shifted along the X axis towards the origin of coordinates, while P_3 is shifted in the opposite direction until it reaches the position in which $x^{(3)} = 1/3$. The case $x^{(3)} = 1/3$ occurs on the straight line α_1, which is given by the equation $2\mu + 3\sigma = 0$. A qualitative change in the phase portrait of the system then occurs. When $2\mu + 3\sigma = 0$, solution (2.11) describes a single-parameter family of equilibrium positions, which lie on a circle with centre at the origin of coordinates and radius $\sqrt{2J_0/3}$, that bounds the region of possible motion of the system (see Fig. 3a). The phase trajectory (the separatrice), which, as $\tau \to \pm\infty$ asymptotically approaches the equilibrium position $R_2 = J_0/3$, $\theta = \pm\pi/2$ (points P_4 and P_5 in Fig. 3a), devides the phase portrait into two subregions, the motion in which occurs along a

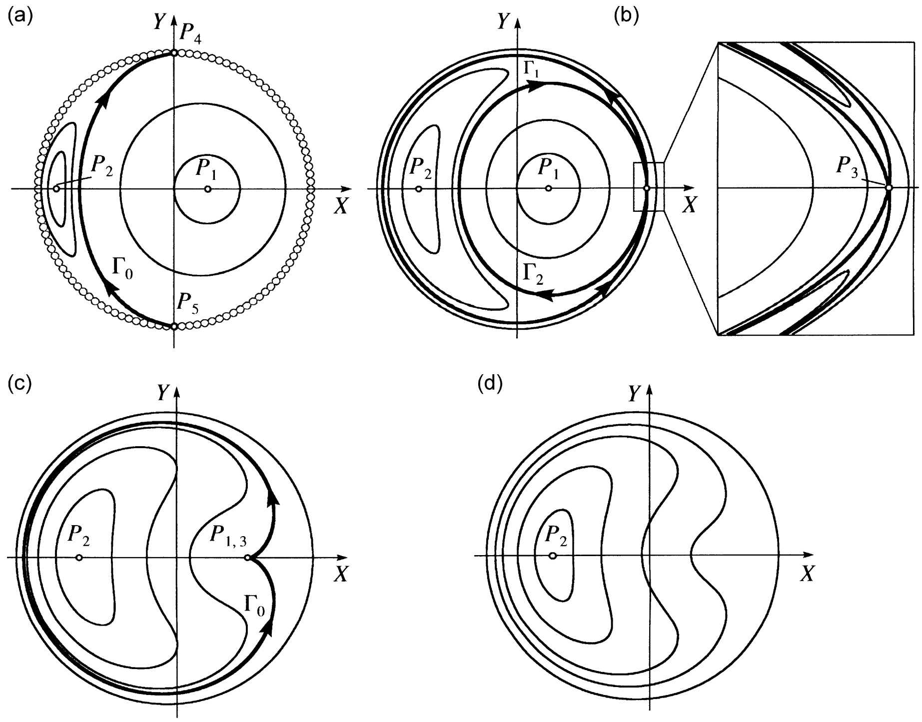

Fig. 3.

closed curve, that envelops stable equilibrium positions P_1 and P_2. On passing through the straight line α_1 the value of θ, corresponding to the root $x^{(3)}$, changes from π to 0, i.e., the equilibrium position P_3 is now situated in the half-plane $X>0$ (Fig. 3b). Note that, in the truncated system with Hamiltonian (2.1), the sets of short-period and long-period motions on the straight line α_1 coincide, which leads to orbital instability of the short-period motions.

When the values of the parameters approach curves β_2, the equilibrium positions P_1 and P_3 approach one another and merge on the curve β_2, forming an unstable equilibrium position $P_{1,3}$ (Fig. 3c), which disappears on approaching subregion III (Fig. 3d).

The reduced system can be integrated in Jacobi elliptic functions. The form of its general solution depends very much on the values of the parameters σ, μ and the constants J_0 and h. We will consider this question in more detail.

Using the integral $F=\varepsilon h$ (see formula (2.5)), we eliminate the variable θ from the second equation of system (2.4) and arrive at the following equation for R_2

$$\frac{dR_2}{d\tau} = \mp\varepsilon\sqrt{R_2(J_0+3R_2)^3-(\sigma J_0 R_2+\mu R_2^2-h)^2} \tag{2.12}$$

The upper and lower signs correspond to motion in the half-planes $Y<0$ and $Y>0$ (see Figs 2 and 3), where the variable R_2 decreases and increases respectively.

We will denote by $h^{(i)}$ the value of the constant h in equilibrium position (2.6), corresponding to the real root $x^{(i)}$ ($i=1, 2, 3$) of Eq.(2.7), and by h_* the value of h in solution (2.11). We have from relations (2.4) and (2.5)

$$h^{(i)} = \frac{J_0^2 x^{(i)}(\sigma+6x^{(i)}\sigma+3\mu x^{(i)})}{12x^{(i)}-1}, \quad i = 1, 2, 3, \quad h_* = \frac{J_0^2}{9}(\mu+3\sigma) \tag{2.13}$$

The values of the constant h, for which motion of system (2.4) is possible, always lie in a limited interval and are defined as follows:

$$h \in \begin{cases} [h_*, h^{(1)}], & \text{если } \mu<\tilde{\mu}_- \\ [h^{(2)}, h^{(1)}], & \text{если } \tilde{\mu}_- \le \mu < \tilde{\mu}_+; \quad \tilde{\mu}_\pm = -\frac{9}{4}\sigma \pm \frac{3}{4}\sqrt{\sigma^2+6} \\ [h^{(2)}, h_*], & \text{если } \mu \ge \tilde{\mu}_+ \end{cases} \tag{2.14}$$

The behaviour of the phase trajectories and the form of the corresponding solution of system (2.4) depend very much on the roots of the polynomial

$$G(x) = x(1-3x)^3-(\sigma x+\mu x^2-c)^2; \quad c = h/J_0^2 \tag{2.15}$$

Note that the polynomial $G(x)$ only has multiple roots when $h=h_*$ and $h=h^{(i)}$ ($i=1, 2, 3$). These values of the constant h divide the region of possible motion into several intervals, in which polynomial (2.15) has a fixed number of simple real roots; it can be equal to two or four.

Suppose h takes values from the interval where the polynomial $G(x)$ has four real roots $x_1 < x_2 < x_3 < x_4$. We will initially put $R_2(0) = J_0 x_1$, in which case we have from Eq. (2.12)

$$R_2(\tau) = J_0 \frac{x_1(x_4 - x_2) + x_4(x_2 - x_1)\operatorname{sn}^2(u, k)}{x_4 - x_2 + (x_2 - x_1)\operatorname{sn}^2(u, k)} \tag{2.16}$$

where

$$k = \sqrt{\frac{(x_4 - x_3)(x_2 - x_1)}{(x_4 - x_2)(x_3 - x_1)}}, \quad u = \frac{1}{2}\varepsilon J_0 \sqrt{(27 + \mu^2)(x_4 - x_2)(x_3 - x_1)}\tau \tag{2.17}$$

The motion corresponding to solutions (2.16) is bounded and satisfies the inequality

$$J_0 x_1 \le R_2 \le J_0 x_2$$

We now put $R_2(0) = J_0 x_4$, and we then obtain from Eq. (2.12)

$$R_2(\tau) = J_0 \frac{x_4(x_3 - x_1) + x_1(x_4 - x_3)\operatorname{sn}^2(u, k)}{x_3 - x_1 + (x_4 - x_3)\operatorname{sn}^2(u, k)} \tag{2.18}$$

where we have used notation (2.17) for k and u. The following inequality is satisfied on the motion corresponding to solution (2.18),

$$J_0 x_3 \le R_2 \le J_0 x_4$$

Note that solutions (2.16) and (2.18) describe the change in the variable R_2 on two different trajectories of the reduced system, which correspond to the same value of h. When $k \ne 1$ these trajectories correspond to periodic motions with frequency

$$\omega = \frac{\varepsilon\pi J_0 \sqrt{(27 + \mu^2)(x_4 - x_2)(x_3 - x_1)}}{2\mathbf{K}(k)} \tag{2.19}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

Suppose now that h belongs to the range of values for which the polynomial $G(x)$ has two real roots $x_1 < x_2$ and two complex roots $x_3 = y - iz$, $x_4 = y + iz$. In this case, assuming $R_2(0) = J_0 x_1$, we have the following solution of Eq. (2.12)

$$R_2(\tau) = J_0 \frac{x_1 P + x_2 Q + (x_1 P - x_2 Q)\operatorname{cn}(u, k)}{P + Q + (P - Q)\operatorname{cn}(u, k)} \tag{2.20}$$

where

$$Q = \sqrt{(y - x_1)^2 + z^2}, \qquad P = \sqrt{(y - x_2)^2 + z^2}$$
$$k = \sqrt{\frac{(x_2 - x_1)^2 - (P - Q)^2}{4QP}}, \quad u = \varepsilon J_0 \sqrt{(27 + \mu^2)QP}\tau \tag{2.21}$$

When $k \ne 1$ solution (2.20) is periodic and describes bounded motion in the range $J_0 x_1 \le R_2 \le J_0 x_2$. The frequency of the corresponding oscillations is calculated from the formula

$$\omega = \frac{\varepsilon\pi J_0 \sqrt{(27 + \mu^2)QP}}{2\mathbf{K}(k)} \tag{2.22}$$

The limit case $k = 0$ is only possible when $h = h_*$ or $h = h^{(i)}$ ($i = 1, 2$), and the limit case $k = 1$ occurs when $h = h^{(3)}$. In these cases formulae (2.16), (2.18) and (2.20) are also applicable. When $k = 1$ they describe asymptotic motions on the homoclinics Γ_1 and Γ_2 (see Figs 2 and 3).

Using formulae (2.16), (2.18) and (2.20) and the first integral $F = \varepsilon h$, we can obtain an explicit expression for θ as a function of τ and complete the integration of system (2.4). When the solution of the reduced system is obtained, integration of the truncated system with Hamiltonian (2.1) reduces to the calculation of a quadrature.

Hence, depending on the initial conditions, the solutions of the truncated system describe either periodic motions or motions that are asymptotic to the periodic motions, or conditionally periodic motions with frequencies $\Omega = 1 + O(\varepsilon)$ and ω, where ω is specified by formulae (2.19) or (2.22). The periodic motions occur along closed trajectories (periodic orbits) of the phase space of the truncated system. The trajectories that are asymptotic to the unstable periodic orbits form invariant manifolds, which divide the phase space of the truncated system into regions of conditionally periodic motion.

3. The motions of the whole system

Many properties of the motions of the truncated system are also preserved in a small neighbourhood of the equilibrium position of the whole system. In particular, in the whole system there are families of periodic orbits, similar to the periodic orbits of the truncated system.[6] We will show that the conditionally periodic trajectories of the whole system also correspond to the majority of the conditionally periodic trajectories of the truncated system.

We will introduce the action-angle variables I_i, ω_i ($i = 1, 2$) into the regions of conditionally periodic motions of the phase space of the truncated system. Hamiltonian (2.3) of the truncated system does not depend explicitly on the coordinate ψ_1, and hence we can put $I_1 = J$. We will introduce the second variable (the action I_2) given by the formula (see, for example, Ref. 21)

$$I_2(h, I_1) = \frac{1}{2\pi}\oint R_2 d\theta \tag{3.1}$$

The integral is evaluated along the closed curve $F = \varepsilon h$ of the phase plane of the reduced system. Suppose $R_2^{(1)}$ and $R_2^{(2)}$ are the maximum and minimum values of the variable R_2 on this curve. Taking into account the fact that the variable R_2 takes its extremal values $R_2^{(1)}$ and $R_2^{(2)}$ when $\theta = 0$ or $\theta = \pi$, and also using Eq. (2.4) and the first integral $F = \varepsilon h$, we can rewrite equality (3.1) in the form

$$I_2(h, I_1) = \frac{1}{2\pi}\int_{R_2^{(1)}}^{R_2^{(2)}} \frac{2R_2(\sigma J + 2\mu R_2)(J - 3R_2) + (J - 12R_2)(h - \sigma J R_2 - \mu R_2^2)}{(J - 3R_2)\sqrt{R_2(J - 3R_2)^3 - (h - \sigma J R_2 - \mu R_2^2)^2}} dR_2 \tag{3.2}$$

Inversion of (3.2) gives $h = h(I_1, I_2)$ and enables us to obtain the Hamiltonian F of the reduced system in I_1, I_2 variables

$$F(\theta, R_2) = \varepsilon h(I_1, I_2) \tag{3.3}$$

Then, the canonical transformation

$$\psi_1, \theta, J, R_2 \to \omega_1, \omega_2, I_1, I_2$$

can be specified by the following generating function (see Ref. 8)

$$S(I_1, I_2, \psi_1, \theta) = I_1\psi_1 + \int_0^\theta R_2 d\theta \tag{3.4}$$

where $R_2 = R_2(I_1, I_2, \theta)$ is the function defined by relation (3.3).

In the action-angle variables the Hamiltonian of the truncated system takes the form

$$H_* = H^{(0)}(I_1) + \varepsilon H^{(1)}(I_1, I_2), \quad H^{(0)}(I_1) = I_1 \text{ и } H^{(1)}(I_1, I_2) = h(I_1, I_2) + a_{02}I_1^2 \tag{3.5}$$

We will consider the motion in the complete system. In I_i, ω_i variables ($i = 1, 2$), Hamiltonian (1.3) has the form

$$H = H^{(0)}(I_1) + \varepsilon H^{(1)}(I_1, I_2) + \varepsilon^2 H^{(2)}(I_1, I_2, \omega_1, \omega_2, \varepsilon^{1/2}) \tag{3.6}$$

The function (3.6) depends analytically on all its arguments and depends 2π-periodically on ω_1 and ω_2.

Since, when $\varepsilon = 0$, Hamiltonian (3.6) depends only on one variable I_1, we have the case of inherent degeneracy.[22] It was shown in Refs 22 and 23 that if the inequalities

$$\frac{\partial H^{(0)}}{\partial I_1} \neq 0, \quad \frac{\partial H^{(1)}}{\partial I_2} \neq 0, \quad \frac{\partial^2 H^{(1)}}{\partial I_2^2} \neq 0 \tag{3.7}$$

are satisfied in the motions of the truncated system, then for the majority of initial conditions, the motions of the complete system are conditionally periodic motions. In addition, for all initial conditions the action variables remain for an infinitely long time in the region of its initial values[22] so that we have the following estimate[23]

$$|I_i(t) - I_i(0)| < d_1\varepsilon^{1/2}, \quad d_1 = \text{const} \tag{3.8}$$

The trajectories of the complete system, which are not conditionally periodic, belong to a set of exponentially small measure of the order of $\exp(-d_2/\varepsilon^{\frac{1}{2}})$, where $d_2 = \text{const} > 0$.

We will verify that inequalities (3.7) are satisfied. The first and second of these denote that the frequencies

$$\Omega = \frac{\partial H^{(0)}}{\partial I_1} + O(\varepsilon), \quad \omega = \varepsilon\frac{\partial H^{(1)}}{\partial I_2} \tag{3.9}$$

of the conditionally periodic motions of the truncated system are not equal to zero. Hence, the first of inequalities (3.7) is obviously satisfied. It follows from formulae (2.19) and (2.22) that $\omega = 0$ either when $k = 1$, which corresponds to asymptotic trajectories, or when $J_0 = 0$, i.e., in an equilibrium position. Hence, in conditionally periodic motions of the truncated system the second of inequalities (3.7) is also satisfied.

In order to verify the third inequality of (3.7) we will use the identity

$$\frac{\partial^2 H^{(1)}}{\partial I_2^2} = \frac{1}{2\varepsilon^2}\frac{d\omega^2}{dh} \tag{3.10}$$

Bearing in mind formulae (2.19) and (2.22), and also using the relations for complete elliptic integrals of the first and second kind, equality (3.10) can be written in explicit form. Omitting the lengthy algebra, we will write the final result.

Two cases are possible depending on the roots of Eq. (2.15). If Eq. (2.15) has four real roots x_l ($i = 1, 2, 3, 4$), then

$$\frac{\partial^2 H^{(1)}}{\partial I_2^2} = \frac{\pi^2 I_1^2 (A_1 + A_2 + A_3 + A_4)}{4k^2 \mathbf{K}^3(k)} \tag{3.11}$$

where

$$\begin{aligned} A_1 &= \frac{(x_4 - x_3)((x_4 - x_2)\mathbf{E}(k) - (x_4 - x_1)\mathbf{K}(k))\xi_1}{(x_2 - x_1)(x_4 - x_1)^2(x_3 - x_1)} \\ A_2 &= \frac{(x_4 - x_3)((x_3 - x_1)\mathbf{E}(k) - (x_3 - x_2)\mathbf{K}(k))\xi_2}{(x_2 - x_1)(x_3 - x_2)^2(x_4 - x_2)} \\ A_3 &= \frac{(x_2 - x_1)((x_4 - x_2)\mathbf{E}(k) - (x_3 - x_2)\mathbf{K}(k))\xi_3}{(x_4 - x_3)(x_3 - x_2)^2(x_3 - x_1)} \\ A_4 &= \frac{(x_2 - x_1)((x_3 - x_1)\mathbf{E}(k) - (x_4 - x_1)\mathbf{K}(k))\xi_4}{(x_4 - x_3)(x_4 - x_1)^2(x_4 - x_2)} \end{aligned} \tag{3.12}$$

If Eq. (2.15) has two real roots x_1 and x_2 and two complex roots $x_3 = y - iz$ and $x_4 = y + iz$, then

$$\frac{\partial^2 H^{(1)}}{\partial I_2^2} = \frac{\pi^2 I_1^2 (B_1 + B_2 + B_3 + B_4)}{64 P^2 Q^2 k^2 (1 - k^2)\mathbf{K}^3(k)} \tag{3.13}$$

where

$$\begin{aligned} B_1 &= \frac{(x_3 - x_4)^2 (PQ\mathbf{K}(k) + (x_2 - x_4)(x_2 - x_3)(\mathbf{K}(k) - 2\mathbf{E}(k)))\xi_1}{(x_1 - x_4)(x_1 - x_3)} \\ B_2 &= \frac{(x_3 - x_4)^2 (PQ\mathbf{K}(k) + (x_1 - x_4)(x_1 - x_3)(\mathbf{K}(k) - 2\mathbf{E}(k)))\xi_2}{(x_2 - x_4)(x_2 - x_3)} \\ B_3 &= \frac{(x_1 - x_2)^2 (PQ\mathbf{K}(k) + (x_1 - x_4)(x_2 - x_4)(\mathbf{K}(k) - 2\mathbf{E}(k)))\xi_3}{(x_2 - x_3)(x_1 - x_3)} \\ B_4 &= \frac{(x_1 - x_2)^2 (PQ\mathbf{K}(k) + (x_1 - x_3)(x_2 - x_3)(\mathbf{K}(k) - 2\mathbf{E}(k)))\xi_4}{(x_2 - x_4)(x_1 - x_4)} \end{aligned} \tag{3.14}$$

Here

$$\xi_i = \sigma x_i + \mu x_i^2 - c, \quad i = 1, 2, 3, 4$$

The quantities P, Q and c are found from the formulae in Section 2. The complete elliptic integrals of the first and second kind are denoted by $\mathbf{K}(k)$ and $\mathbf{E}(k)$ respectively. In relations (3.11) and (3.12) the quantity k is calculated from formula (2.17), while in (3.13) and (3.14) it is calculated from formula (2.21). Since the roots x_3 and x_4 are complex conjugates, the expresions for B_1 and B_2 are real while the expressions for B_3 and B_4 are complex conjugates. Hence, expression (3.13) only takes real values.

Expressions (3.11) and (3.13) depend on the parameters σ and μ and the constant c. On the basis of formulae (3.13)–(3.14) the third inequality of (3.7) was verified numerically for arbitrary values of σ, μ and c. Calculations show that this inequality is satisfied for any values of the parameters.

Since conditions (3.7) are satisfied, then, in the ε-neighbourhood of an equilibrium position, the majority of trajectories of the complete system are conditionally periodic. We then have estimate (3.8). Hence, for any initial conditions the motion in the ε-neighbourhood of an equilibrium position is bounded. It follows from the latter that instability of the periodic orbits, which start from an equilibrium position, can only have a local character. This means that the trajectories beginning in the region of an unstable periodic orbit, remain within a certain bounded neighbourhood of it as long as desired.

4. The motions of a satellite in the region of its regular precession

We will consider the dynamics of a symmetrical satellite – a rigid body, the centre of mass O of which moves in a circular orbit. Suppose $Oxyz$ is a connected system of coordinates, the axes of which are directed along the principal central axes of inertia of the satellite (the Oz axis coincides with the axis of dynamic symmetry). We will direct the axes of an orbital system of coordinates $OXYZ$ along the radius vector of the centre of mass (OZ), the transversal (OX) and the binormal (OY) to the orbit. We will specify the orientation of the connected system of coordinates with respect to the orbital system using the Euler angles ψ, θ, φ.

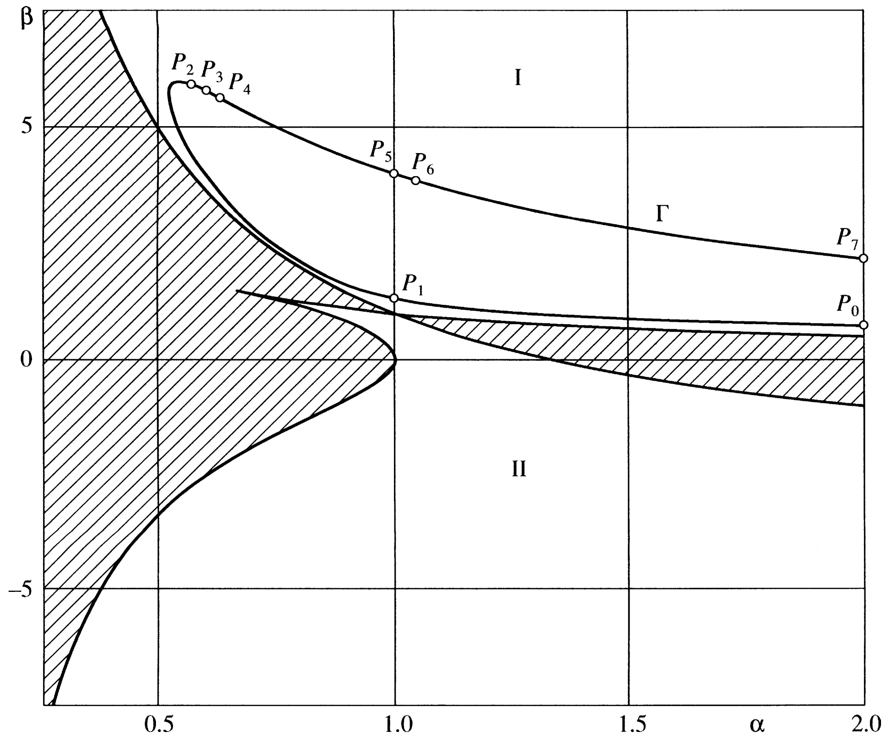

Fig. 4.

The equations of motion of the satellite about the centre of mass can be written in canonical form with Hamiltonian[13]

$$H = \frac{p_\psi^2}{2\sin^2\theta} + \frac{p_\theta^2}{2} - p_\psi \operatorname{ctg}\theta\cos\psi - \alpha\beta p_\psi\frac{\cos\theta}{\sin^2\theta} - p_\theta\sin\psi + \alpha\beta\frac{\cos\psi}{\sin\theta} +$$
$$+ \frac{\alpha^2\beta^2}{2\sin^2\theta} + \frac{3}{2}(\alpha - 1)\cos^2\theta \tag{4.1}$$

The dimensionless momenta, corresponding to the coordinates ψ, θ, φ are denoted p_ψ, p_θ, p_φ. The angle φ is the cyclic coordinate, and hence in (4.1) we have put $p_\psi = \alpha\beta = \text{const}$, where $\alpha = C/A$ and $\beta = \Omega_0/\omega_0$, A, B and C $(A = B)$ are the principal central moments of inertia, and ω_0 and Ω_0 are the average motion of the centre of mass in the orbit and the projection of the absolute angular velocity of the satellite onto its axis of dynamic symmetry, respectively. The parameter α takes values in the interval [0, 2], while the parameter β can be any real number.

The canonical system with Hamiltonian (4.1) has the particular solution[24]

$$\psi = \pi, \quad \theta = \pi/2, \quad p_\psi = 0, \quad p_\theta = 0 \tag{4.2}$$

that describes cylindrical precession, which is the steady rotation of the satellite about its axis of dynamic symmetry, situated perpendicular to the orbital plane. The problem of the stability of the cylindrical precession was investigated in detail in Refs 13, 25-29. In Fig. 4 the instability region is shown hatched. In the stability region *I* the quadratic part of the Hamiltonian of the equations of perturbed motion is positive definite, while the curve Γ situated in this region, corresponds to 3:1 resonance. On the basis of the results obtained in Sections 2 and 3, we can carry out a non-linear analysis of the motions of the satellite in the neighbourhood of its cylindrical precession for values of the parameters pertaining to the resonance curve Γ.

The points separating the curve Γ into parts, for which the motion of the satellite in the neighbourhood of cylindrical precession has a qualitatively different character, are as follows:

$$P_0(2, 0.73074), \quad P_1(1, 4/3), \quad P_2(0.59219, 5.82501), \quad P_3(0.61460, 5.71177)$$

$$P_4(0.64446, 5.55217), \quad P_5(1, 4), \quad P_6(1.06066, 3.81192), \quad P_7(2, 2.19836)$$

Calculations show that for values of the parameters pertaining to the parts (P_1, P_2), (P_4, P_5) and (P_6, P_7) of the curve Γ only one family of long-period motions exist, originating from the cylindrical precession of the satellite. Here, on the part (P_1, P_2) the conditionally periodic motions in the neighbourhood of the cylindrical precession correspond to the phase portrait in Fig. 2a, and on the parts (P_4, P_5) and (P_6, P_7) they correspond to the phase portraits in Fig. 3d. For values of the parameters belonging to the sections (P_0, P_1), (P_2, P_3), (P_3, P_4) and (P_5, P_6) of curve Γ, there are three families of long-period motions, originating from the cylindrical precession of the satellite. On the parts (P_0, P_1), (P_3, P_4) and (P_5, P_6) the conditionally periodic motions in the neighbourhood of the cylindrical precession correspond to the phase portraits in Fig. 3b, while on the part (P_2, P_3) they correspond to the phase portrait in Fig. 2c. Note that one of the three families of long-period motions is unstable. However, this instability is only local, i.e., motions beginning close to the unstable periodic orbit remain infinitely long in a certain limited neighbourhood of it. At the points P_2, P_3, P_4 and P_6 bifurcation of the periodic motions occurs. In this case the conditionally periodic motions in the neighbourhood of cylindrical precession correspond to the phase portraits in Fig. 2b, in Fig. 3a for the points P_2 and P_3 respectively, and in Fig. 3c for the points P_4 and P_6.

At the points of the resonance curve P_1 and P_5 the results obtained above are inapplicable, since the coefficient b of the normal form (1.2) vanishes (the resonance part disappears). At these points $\alpha = 1$, i.e., the satellite is spherically symmetrical and the equations of motion can be integrated explicitly.

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